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The use of spacings in the estimation of a scale parameter ☆

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Abstract

Linear functions on spacings—instead of linear functions on order statistics—are considered, in order to simplify the form of best linear unbiased estimators (BLUEs) and best linear invariant estimators (BLIEs) for the scale parameter in the classical location-scale family. Also, a sufficient condition for the non-negativity of the scale estimator is presented and, moreover, necessary and sufficient conditions for the BLUE (and the BLIE) to be a constant multiple of the sample range are derived. Finally, a modification of this approach is applied in order to simplify the derivations of both the location and the scale estimators in the Uniform Type-II Censored model.

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1. Introduction

In this article we are mainly concerned with the estimation of the scale parameter θ_2 in the classical location-scale family

$$\{F((\cdot-\theta_1)/\theta_2);\ \theta_1\in\mathbb{R},\theta_2>0\},$$

where $F(\cdot)$ is a known d.f. with positive finite variance (thus, F is non-degenerate). In particular, consider the random sample $X_1^*,X_2^*,\ldots,X_n^*$ from $F((\cdot-\theta_1)/\theta_2)$ and the corresponding ordered

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sample $X^*_{1:n} \leqslant X^*_{2:n}, \leqslant \cdots \leqslant X^*_{n:n}$, and also let $X_1, X_2, \ldots, X_n$ and $X_{1:n} \leqslant X_{2:n}, \leqslant \cdots \leqslant X_{n:n}$ be the corresponding samples from the completely known d.f. $F(\cdot)$. Then, since

$$(X^*_{1:n}, X^*_{2:n}, \ldots, X^*_{n:n})' \stackrel{\mathrm{d}}{=} (\theta_1 + \theta_2 X_{1:n}, \theta_1 + \theta_2 X_{2:n}, \ldots, \theta_1 + \theta_2 X_{n:n})',$$

it follows that any linear estimator L (i.e., a linear function on order statistics) has the form

$$L = \sum_{i=1}^{n} c_i^* X^*_{i:n} \stackrel{\mathrm{d}}{=} \theta_1 \sum_{i=1}^{n} c_i^* + \theta_2 \sum_{i=1}^{n} c_i^* X_{i:n},$$

for some constants c_i^*, $i = 1, 2, \ldots, n$. Therefore, a necessary and sufficient condition for L to be invariant (i.e., independently distributed of the location parameter θ_1) is

$$\sum_{i=1}^{n} c_i^* = 0.$$

Observe that if this is the case, then there exist constants c_i, $i = 1, \ldots, n-1$, such that

$$L = \sum_{i=1}^{n-1} c_i Z_i^* \stackrel{\mathrm{d}}{=} \theta_2 \sum_{i=1}^{n-1} c_i Z_i, \tag{1.1}$$

where $Z_i^* = X^*_{i+1:n} - X^*_{i:n} \stackrel{\mathrm{d}}{=} \theta_2 (X_{i+1:n} - X_{i:n}) = \theta_2 Z_i$, $i = 1, \ldots, n-1$ are, respectively, the spacings from $F((\cdot - \theta_1)/\theta_2)$ and the completely known $F(\cdot)$ (this is an immediate consequence of the fact that

$$(Z_1^*, \ldots, Z_{n-1}^*)' \stackrel{\mathrm{d}}{=} \theta_2 (Z_1, \ldots, Z_{n-1})',$$

which enable us to express L as a linear function on Z_i's, as in (1.1)).

Now, let $\boldsymbol{X} = (X_{1:n}, X_{2:n}, \ldots, X_{n:n})'$ and $\boldsymbol{Z} = (Z_1, \ldots, Z_{n-1})'$ be the random vectors of order statistics and spacings, respectively, from the known d.f. $F(\cdot)$, and use the notation

$$\boldsymbol{\mu} = \mathbb{E}[\boldsymbol{X}], \quad \boldsymbol{\Sigma} = \mathbb{D}[\boldsymbol{X}] \quad \text{and} \quad \boldsymbol{S} = \mathbb{E}[\boldsymbol{X}\boldsymbol{X}'], \tag{1.2}$$

while the corresponding quantities for $\boldsymbol{Z}$ are denoted by

$$\boldsymbol{m} = \mathbb{E}[\boldsymbol{Z}], \quad \boldsymbol{D} = \mathbb{D}[\boldsymbol{Z}] \quad \text{and} \quad \boldsymbol{E} = \mathbb{E}[\boldsymbol{Z}\boldsymbol{Z}'], \tag{1.3}$$

where $\mathbb{D}[\boldsymbol{\zeta}]$ denotes the dispersion matrix of the random vector $\boldsymbol{\zeta}$ (note that the vectors and matrices in (1.2) are of order n, while the corresponding ones in (1.3) are of order $n-1$). Of course,

$$\boldsymbol{\Sigma} = \boldsymbol{S} - \boldsymbol{\mu}\boldsymbol{\mu}', \quad \boldsymbol{\Sigma} > 0, \quad \boldsymbol{S} > 0,$$

and, similarly

$$\boldsymbol{D} = \boldsymbol{E} - \boldsymbol{m}\boldsymbol{m}', \quad \boldsymbol{D} > 0, \quad \boldsymbol{E} > 0.$$

In the present paper we present an effective technique for the derivation of the best linear unbiased estimator (BLUE) and the best linear invariant estimator (BLIE) of θ_2, based on simple properties satisfied by the spacings (Propositions 2.1 and 2.2). This approach, i.e., the use of spacings instead of order statistics, turns out to be much more convenient for theoretical and applied purposes; e.g., it enables us to give an explicit form for the constant $a = \text{BLIE}/\text{BLUE}$ (Lemma 2.1), to find necessary and sufficient conditions for the BLUE of θ_2 to be a constant multiple of the sample range (Theorem

3.1) and, furthermore, to present a sufficient condition for the non-negativity of the scale estimator (Theorem 4.1), that seems to be accurate enough for many cases.

Finally, we discuss a similar approach for the Uniform Type-II Censored model, yielding easily some known results of Sarhan and Greenberg (1959), concerning both the location parameter θ_1 and the scale parameter θ_2 (Section 5 and examples of Section 6).

2. BLUEs and BLIEs

Using the notation given in the introduction, it is well-known (Lloyd (1952); see also Arnold et al. (1992), Chapter 7) that the BLUE of θ_2 is given by

$$L_{\mathrm{U}} = \frac{\mathbf{1}'\boldsymbol{\Sigma}^{-1}(\mathbf{1}\boldsymbol{\mu}' - \boldsymbol{\mu}\mathbf{1}')\boldsymbol{\Sigma}^{-1}\boldsymbol{X}^*}{(\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1})(\boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}) - (\mathbf{1}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu})^2}, \tag{2.1}$$

where $\boldsymbol{X}^* = (X^*_{1:n}, X^*_{2:n}, \ldots, X^*_{n:n})'$ and $\mathbf{1}' = (1, 1, \ldots, 1) \in \mathbb{R}^n$; the variance of L_{U} (which is the MSE of L_{U} since, by construction, it is unbiased for θ_2) is given by

$$\mathrm{Var}[L_{\mathrm{U}}] = \frac{(\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1})\theta_2^2}{(\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1})(\boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}) - (\mathbf{1}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu})^2}. \tag{2.2}$$

In the case where $F(\cdot)$ is symmetric, we take θ_1 throughout to be the mean. Then formulae (2.1) and (2.2) are simplified to the following one:

$$L_{\mathrm{U}} = \frac{\boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\boldsymbol{X}^*}{\boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}} \quad \text{with } \mathrm{Var}[L_{\mathrm{U}}] = \frac{\theta_2^2}{\boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}}. \tag{2.3}$$

Both expressions (2.1) and (2.2) for a general $F(\cdot)$ can be simplified to an expression similar to (2.3), if we use spacings instead of order statistics. In particular, we have the following

Proposition 2.1. *Under the above assumptions and the notation of Section* 1, *the* BLUE *of* θ_2 *and its variance are given by*

$$L_{\mathrm{U}} = \frac{\boldsymbol{m}'\boldsymbol{D}^{-1}\boldsymbol{Z}^*}{\boldsymbol{m}'\boldsymbol{D}^{-1}\boldsymbol{m}} \quad \textit{with } \mathrm{Var}[L_{\mathrm{U}}] = \frac{\theta_2^2}{\boldsymbol{m}'\boldsymbol{D}^{-1}\boldsymbol{m}}, \tag{2.4}$$

where $\boldsymbol{Z}^* = (Z_1^*, \ldots, Z_{n-1}^*)'$.

Proof. Since the form of the most general linear location-invariant estimator of θ_2 is $L = \boldsymbol{c}'\boldsymbol{Z}^*$, where $\boldsymbol{c} = (c_1, \ldots, c_{n-1})'$ (see (1.1)), it follows that it is unbiased for θ_2 iff

$$\boldsymbol{c}'\boldsymbol{m} = 1; \tag{2.5}$$

on the other hand, its variance is given by

$$\mathrm{Var}[L] = (\boldsymbol{c}'\boldsymbol{D}\boldsymbol{c})\theta_2^2. \tag{2.6}$$

Thus, we wish to minimize (2.6) under restriction (2.5). Taking into account the Lagrangian $Q(\boldsymbol{c}; \lambda) = \boldsymbol{c}'\boldsymbol{D}\boldsymbol{c} - 2\lambda(\boldsymbol{c}'\boldsymbol{m})$, it is easily seen that the optimum value is $\boldsymbol{c} = \lambda(\boldsymbol{D}^{-1}\boldsymbol{m})$, and the restriction yields $\lambda = 1/(\boldsymbol{m}'\boldsymbol{D}^{-1}\boldsymbol{m})$; this completes the proof. □

Observing that (2.1) and (2.4) are two different forms *of the same estimator* L_{U}, and equating variances, it follows that

$$\boldsymbol{m}'\boldsymbol{D}^{-1}\boldsymbol{m} = \boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} - \frac{(\mathbf{1}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu})^2}{\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1}} \leqslant \boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu},$$

with equality iff $\mathbf{1}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} = 0$ (this happens for symmetric $F(\cdot)$). This identity holds *for all* d.f.'s with finite strictly positive variance, showing a non-obvious connection between the mean-vectors and the dispersion matrices of $\boldsymbol{Z}$ and $\boldsymbol{X}$. On the other hand, the calculations involved in (2.4) are much simpler than those involved in (2.1).

Giving up the requirement of unbiasedness, Mann (1969) obtained the form of BLIE (i.e., of the Best Linear Invariant Estimator) to be

$$L_{\mathrm{I}} = \left(\boldsymbol{\mu}' - \frac{\mathbf{1}'\boldsymbol{S}^{-1}\boldsymbol{\mu}}{\mathbf{1}'\boldsymbol{S}^{-1}\mathbf{1}}\mathbf{1}'\right)\boldsymbol{S}^{-1}\boldsymbol{X}^*, \tag{2.7}$$

while the corresponding MSE is

$$\mathrm{MSE}[L_{\mathrm{I}}] = \mathbb{E}[L_{\mathrm{I}} - \theta_2]^2 = \left(1 + \frac{(\mathbf{1}'\boldsymbol{S}^{-1}\boldsymbol{\mu})^2}{(\mathbf{1}'\boldsymbol{S}^{-1}\mathbf{1})} - \boldsymbol{\mu}'\boldsymbol{S}^{-1}\boldsymbol{\mu}\right)\theta_2^2. \tag{2.8}$$

In the case of a symmetric population, $\mathbf{1}'\boldsymbol{S}^{-1}\boldsymbol{\mu} = 0$ and (2.7), (2.8) reduce to

$$L_{\mathrm{I}} = \boldsymbol{\mu}'\boldsymbol{S}^{-1}\boldsymbol{X}^* \text{ with } \mathrm{MSE}[L_{\mathrm{I}}] = (1 - \boldsymbol{\mu}'\boldsymbol{S}^{-1}\boldsymbol{\mu})\theta_2^2. \tag{2.9}$$

However, using spacings instead of order statistics, one can easily derive a very simple expression for the BLIE of θ_2 (without imposing symmetry on the population). In fact, the following proposition can be easily established.

Proposition 2.2. *Under the above assumptions and the notation of Section* 1, *the* BLIE *of* θ_2 *and its* MSE *are given by*

$$L_{\mathrm{I}} = \boldsymbol{m}'\boldsymbol{E}^{-1}\boldsymbol{Z}^* \text{ with } \mathrm{MSE}[L_{\mathrm{I}}] = (1 - \boldsymbol{m}'\boldsymbol{E}^{-1}\boldsymbol{m})\theta_2^2. \tag{2.10}$$

The proof follows by the same arguments as in Proposition 2.1, except that we do not have to use restriction (2.5). Since the estimators in (2.7) and (2.10) coincide, it follows that

$$\boldsymbol{m}'\boldsymbol{E}^{-1}\boldsymbol{m} = \boldsymbol{\mu}'\boldsymbol{S}^{-1}\boldsymbol{\mu} - \frac{(\mathbf{1}'\boldsymbol{S}^{-1}\boldsymbol{\mu})^2}{\mathbf{1}'\boldsymbol{S}^{-1}\mathbf{1}} \leqslant \boldsymbol{\mu}'\boldsymbol{S}^{-1}\boldsymbol{\mu}.$$

The above equation gives a connection between the mean vectors and mean-squared matrices of $\boldsymbol{Z}$ and $\boldsymbol{X}$, satisfied by any d.f. F with finite, strictly positive variance. Moreover, the equality is attained for any symmetric F, and in this case we have

$$\boldsymbol{m}'\boldsymbol{E}^{-1}\boldsymbol{m} = \boldsymbol{\mu}'\boldsymbol{S}^{-1}\boldsymbol{\mu} \leqslant 1.$$

Also note that the form of (2.10) (holding for any F) is quite similar to that of (2.9) (which merely holds for symmetric F), showing once again the simplifications one attains using spacings instead of order statistics.

Since the BLIE has minimum MSE among all the linear invariant functions on spacings, while the BLUE has minimum MSE among all the linear invariant functions on spacings that are unbiased for θ_2, it follows that $\mathrm{MSE}[L_\mathrm{I}] \leqslant \mathrm{MSE}[L_\mathrm{U}] = \mathrm{Var}[L_\mathrm{U}]$, showing that (see (2.4) and (2.10))

$$1 - \boldsymbol{m}'\boldsymbol{E}^{-1}\boldsymbol{m} \leqslant \frac{1}{\boldsymbol{m}'\boldsymbol{D}^{-1}\boldsymbol{m}}, \tag{2.11}$$

this yields a similar inequality for order statistics when the population is symmetric, namely,

$$1 - \boldsymbol{\mu}'\boldsymbol{S}^{-1}\boldsymbol{\mu} \leqslant \frac{1}{\boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}}.$$

However, the equality never holds in (2.11); this happens because of the following:

Lemma 2.1. *There exists a constant* $a = a(F)$, $0 < a < 1$, *depending only on* $F(\cdot)$ (i.e., *a is independent of* θ_1 *and* θ_2), *such that*

$$L_\mathrm{I} = a\,L_\mathrm{U}. \tag{2.12}$$

This constant is given by

$$a = \boldsymbol{m}'\boldsymbol{E}^{-1}\boldsymbol{m} = \frac{\boldsymbol{m}'\boldsymbol{D}^{-1}\boldsymbol{m}}{1 + \boldsymbol{m}'\boldsymbol{D}^{-1}\boldsymbol{m}}. \tag{2.13}$$

Proof. First note that $\boldsymbol{E} = \boldsymbol{D} + \boldsymbol{m}\boldsymbol{m}'$ (see Section 1). This, by Theorem 8.9.3 in Graybill (1969), implies that

$$\boldsymbol{E}^{-1} = \boldsymbol{D}^{-1} - \frac{1}{1 + \boldsymbol{m}'\boldsymbol{D}^{-1}\boldsymbol{m}}(\boldsymbol{D}^{-1}\boldsymbol{m})(\boldsymbol{D}^{-1}\boldsymbol{m})'$$

and, thus,

$$\boldsymbol{m}'\boldsymbol{E}^{-1}\boldsymbol{m} = \boldsymbol{m}'\boldsymbol{D}^{-1}\boldsymbol{m} - \frac{(\boldsymbol{m}'\boldsymbol{D}^{-1}\boldsymbol{m})^2}{1 + \boldsymbol{m}'\boldsymbol{D}^{-1}\boldsymbol{m}},$$

proving the second equality in (2.13). Consider now the class of estimators of the form $L_\lambda = \lambda\,L_\mathrm{U}$, $\lambda \in \mathbb{R}$. It is easy to see that they are linear invariant estimators and, moreover, that

$$\mathrm{MSE}[L_\lambda] = \mathbb{E}[\lambda L_\mathrm{U} - \theta_2]^2 = \left(\frac{\lambda^2}{\boldsymbol{m}'\boldsymbol{D}^{-1}\boldsymbol{m}} + (1-\lambda)^2\right)\theta_2^2.$$

Therefore, minimizing the last expression with respect to λ, we get

$$\lambda = \frac{\boldsymbol{m}'\boldsymbol{D}^{-1}\boldsymbol{m}}{1 + \boldsymbol{m}'\boldsymbol{D}^{-1}\boldsymbol{m}} = a \in (0,1).$$

For this value of $\lambda = a$, it follows that

$$\mathrm{MSE}[L_a] = \theta_2^2/(1 + \boldsymbol{m}'\boldsymbol{D}^{-1}\boldsymbol{m}) = (1 - \boldsymbol{m}'\boldsymbol{E}^{-1}\boldsymbol{m})\theta_2^2 = \mathrm{MSE}[L_\mathrm{I}]$$

(for the second equality we used the second equality in (2.13), and the third equality is included in (2.10)); this shows that L_a has the same MSE as L_I, and since L_a is linear and location invariant, it follows by the uniqueness of L_I that $L_\mathrm{I} = L_a$ with probability 1. This proves both (2.12) and (2.13). □

It should be noted that the assertion that L_I is a constant multiple of L_U is implicitly included in Mann's (1969) results; however, it seems that the form of (2.13) is new.

3. When is the BLUE (or BLIE) of θ_2 a multiple of the sample range?

Bondesson (1976) proved that the BLUE of θ_1 is the sample mean (for all sample sizes $n \geqslant 2$) iff F is a Normal or shifted Gamma (or negative Gamma) distribution. In this section, we find necessary and sufficient conditions under which the BLUE (or, equivalently, the BLIE) of θ_2 is a constant multiple of the sample range

$$R^* = Z_1^* + \cdots + Z_{n-1}^* = X_{n:n}^* - X_{1:n}^*.$$

In particular, we shall prove the following:

Theorem 3.1. *The* BLUE (or the BLIE) *of* θ_2 *is a multiple of the sample range* R^* *iff any one of the following equivalent conditions hold*:

(i) *There exists a constant* λ_1 *such that*

$$\boldsymbol{m} = \lambda_1(\boldsymbol{E}\mathbf{1}).$$

(ii) *There exists a constant* λ_1 *such that*

$$\mathbb{E}[Z_i] = \lambda_1 \mathbb{E}[Z_i R], \quad i = 1, \ldots, n-1,$$

where $R = Z_1 + \cdots + Z_{n-1} = X_{n:n} - X_{1:n}$ *is the sample range of the random sample* $X_1, X_2, \ldots, X_n$ *from the known d.f.* $F(\cdot)$.

(iii) *There exists a constant* λ_2 *such that*

$$\boldsymbol{m} = \lambda_2(\boldsymbol{D}\mathbf{1}).$$

(iv) *There exists a constant* λ_2 *such that*

$$\mathbb{E}[Z_i] = \lambda_2 \operatorname{Cov}[Z_i, R], \quad i = 1, \ldots, n-1,$$

where R *is as in* (ii).

When (i)–(iv) *hold*, *the* BLUE *of* θ_2 *is given by*

$$L_{\mathrm{U}} = \frac{1}{\lambda_1(\mathbf{1}'\boldsymbol{E}\mathbf{1})}(X_{n:n}^* - X_{1:n}^*) = \frac{1}{\lambda_2(\mathbf{1}'\boldsymbol{D}\mathbf{1})}(X_{n:n}^* - X_{1:n}^*)$$

and the BLIE of θ_2 by

$$L_{\mathrm{I}} = \lambda_1(X_{n:n}^* - X_{1:n}^*) = \frac{\lambda_2}{1 + \lambda_2^2(\mathbf{1}'\boldsymbol{D}\mathbf{1})}(X_{n:n}^* - X_{1:n}^*).$$

Proof. First observe that (i) is equivalent to (ii), and (iii) is equivalent to (iv). If (ii) holds, then

$$\mathbb{E}[Z_i](1 - \lambda_1 \mathbb{E}[R]) = \lambda_1 \operatorname{Cov}[Z_i, R], \quad i = 1, \ldots, n-1,$$

which can be rewritten as

$$\mathbb{E}[Z_i] = \lambda_2 \operatorname{Cov}[Z_i, R], \quad i = 1, \ldots, n-1,$$

where $\lambda_2 = \lambda_1/(1 - \lambda_1 \mathbb{E}[R])$, and thus (iv) holds (observe that $\lambda_1 = \mathbb{E}[R]/\mathbb{E}[R^2]$, since

$$\mathbb{E}[R] = \sum_{i=1}^{n-1} \mathbb{E}[Z_i] = \sum_{i=1}^{n-1} \lambda_1 \mathbb{E}[Z_i R] = \lambda_1 \mathbb{E}[R^2]$$

and hence, $0 < \lambda_1 \mathbb{E}[R] < 1$). Conversely, if (iv) holds, then

$$\mathbb{E}[R] = \sum_{i=1}^{n-1} \mathbb{E}[Z_i] = \lambda_2 \sum_{i=1}^{n-1} \mathrm{Cov}[Z_i, R] = \lambda_2 \mathrm{Var}[R],$$

showing that $\lambda_2 = \mathbb{E}[R]/\mathrm{Var}[R] \in (0, +\infty)$, and

$$\mathbb{E}[Z_i] + \lambda_2 \mathbb{E}[Z_i]\mathbb{E}[R] = \lambda_2 \mathbb{E}[Z_i R], \quad i = 1, \ldots, n-1;$$

the last expression can be rewritten as

$$\mathbb{E}[Z_i] = \lambda_1 \mathbb{E}[Z_i R], \quad i = 1, \ldots, n-1,$$

with $\lambda_1 = \lambda_2/(1 + \lambda_2 \mathbb{E}[R])$, which is (ii). Therefore, all conditions (i)–(iv) are equivalent.

Assume now that (i) holds. Then, from (2.10), the BLIE of θ_2 is

$$L_{\mathrm{I}} = \boldsymbol{m}'\boldsymbol{E}^{-1}\boldsymbol{Z}^* = \lambda_1(\mathbf{1}'\boldsymbol{Z}^*) = \lambda_1(X^*_{n:n} - X^*_{1:n}),$$

and the other formulae follow from (2.13) and (2.4). In order to prove necessity, assume that $L_\lambda = \lambda R^* = \lambda(\mathbf{1}'\boldsymbol{Z}^*)$ is the BLIE of θ_2. Then $\mathbb{E}[\lambda R^* - \theta_2]^2 = (\lambda^2(\mathbf{1}'\boldsymbol{E}\mathbf{1}) + 1 - 2\lambda(\mathbf{1}'\boldsymbol{m}))\theta_2^2$ must be minimum with respect to λ, showing that

$$\lambda = \frac{\mathbf{1}'\boldsymbol{m}}{\mathbf{1}'\boldsymbol{E}\mathbf{1}}.$$

Since for this value of λ we must have

$$\mathbb{E}[\lambda R^* - \theta_2]^2 = \left(1 - \frac{(\mathbf{1}'\boldsymbol{m})^2}{\mathbf{1}'\boldsymbol{E}\mathbf{1}}\right)\theta_2^2 = \mathrm{MSE}[L_{\mathrm{I}}] = (1 - \boldsymbol{m}'\boldsymbol{E}^{-1}\boldsymbol{m})\theta_2^2,$$

we conclude that

$$(\mathbf{1}'\boldsymbol{m})^2 = (\mathbf{1}'\boldsymbol{E}\mathbf{1})(\boldsymbol{m}'\boldsymbol{E}^{-1}\boldsymbol{m}).$$

This is the Cauchy–Schwarz inequality written as an equality and, therefore, this equality is attained only if there exists a constant λ_1 such that $\boldsymbol{m} = \lambda_1(\boldsymbol{E}\mathbf{1})$, completing the proof. □

4. Are the scale estimators always non-negative?

Arnold et al. (1992, p. 174), observed that the existence and uniqueness of the BLUE for θ_2 do not guarantee that it is always non-negative. Regarding this question, a particular positive answer (in a more general setting) was given by Bai et al. (1997). Specifically, they proved that if F has a log-concave density f, then the BLUE of θ_2 is positive with probability 1. Our results using spacings, however, enable us to provide a simpler and stronger positive answer to many situations. In this direction, first observe that since $\boldsymbol{Z}^* \geqslant \mathbf{0}$ componentwise, where $\mathbf{0} = (0, \ldots, 0)' \in \mathbb{R}^{n-1}$, the assertion that the BLUE (equivalently, the BLIE) of θ_2 is non-negative is equivalent to the fact that $\boldsymbol{D}^{-1}\boldsymbol{m} \geqslant \mathbf{0}$ componentwise or, equivalently, $\boldsymbol{E}^{-1}\boldsymbol{m} \geqslant \mathbf{0}$ componentwise; this follows from (2.4), (2.10) and Lemma 2.1, since $\boldsymbol{m} = \mathbb{E}[\boldsymbol{Z}] > \mathbf{0}$ componentwise. Therefore, it is easy to prove the following result.

Table 1

z_2 / z_1	0	1	3	4
0	2199/3375	39/3375	3/3375	39/3375
1	507/3375	0	0	0
3	3/3375	78/3375	0	0
4	507/3375	0	0	0

Theorem 4.1. *If either $n=2$ or the known d.f. $F(\cdot)$ is such that*

$$\mathrm{Cov}[Z_i, Z_j] \leqslant 0 \tag{4.1}$$

for all $i \neq j$, $i, j = 1, \ldots, n-1$, then the BLUE (*and the* BLIE) *of θ_2 is non-negative.*

Proof. If $n=2$, then by Theorem 3.1 the BLUE of θ_2 is a multiple of the sample range and the result is obvious. Under (4.1), the positive definite matrix $\boldsymbol{D}$ has non-positive off-diagonal elements. Therefore, from Lemma 2.2 in Bai et al. (1997) (cf. Theorem 12.2.9 in Graybill (1969)) it follows that the positive definite matrix $\boldsymbol{D}^{-1}$ has all its elements non-negative; this shows that $\boldsymbol{D}^{-1}\boldsymbol{m} > \boldsymbol{0}$ componentwise, and the assertion follows from expression (2.4). □

It should be noted that the conclusion of Theorem 4.1 is stronger (in the particular setting of the present article) than the main result of Bai et al. (1997), because of their important by-product:

A log-concave density has negatively correlated spacings.

Note also that most of the distributions that are commonly used in the location-scale families satisfy (4.1); this is not always the case, however, as the following example shows (see also the Pareto d.f. discussed in Section 4 of Bai et al., 1997).

Example 4.1. Let $n=3$ and consider the d.f. $F(\cdot)$ assigning probabilities $\frac{1}{15}$, $\frac{1}{15}$ and $\frac{13}{15}$ to the values 0, 3 and 4, respectively. It follows that the joint probability mass function $\mathbb{P}[Z_1 = z_1, Z_2 = z_2]$ of $(Z_1, Z_2)'$ is given by Table 1. From this we get $\mathbb{E}[Z_1] = 926/1125$, $\mathbb{E}[Z_2] = 94/1125$, $\mathbb{E}[Z_1 Z_2] = 26/375$, $\mathbb{E}[Z_1^2] = 3116/1125$ and $\mathbb{E}[Z_2^2] = 256/1125$, and thus

$$\mathrm{Cov}[Z_1, Z_2] = \frac{706}{(1125)^2} > 0.$$

Nevertheless, the necessary and sufficient condition for L_{U} to be non-negative, i.e., $\boldsymbol{E}^{-1}\boldsymbol{m} \geqslant \boldsymbol{0}$ componentwise, can be rewritten (for $n=3$) as

$$\frac{\mathbb{E}[Z_1 Z_2]}{\mathbb{E}[Z_1^2]} \leqslant \frac{\mathbb{E}[Z_2]}{\mathbb{E}[Z_1]} \leqslant \frac{\mathbb{E}[Z_2^2]}{\mathbb{E}[Z_1 Z_2]},$$

which is also satisfied in this case.

5. Linear estimation for the uniform censored model

Let us assume that $U^*_{1:n} \leqslant U^*_{2:n} \leqslant \cdots \leqslant U^*_{s:n}$ $(2 \leqslant s \leqslant n)$ is a Type-II right censored sample from Uniform $(\theta_1, \theta_1 + \theta_2)$ distribution. Then, since

$$(U^*_{1:n}, U^*_{2:n}, \ldots, U^*_{s:n})' \stackrel{\mathrm{d}}{=} (\theta_1 + \theta_2 U_{1:n}, \theta_1 + \theta_2 U_{2:n}, \ldots, \theta_1 + \theta_2 U_{s:n})',$$

where $U_{1:n} \leqslant U_{2:n} \leqslant \cdots \leqslant U_{s:n}$ is a Type-II right censored ordered sample from the standard Uniform $(0,1)$ d.f., it is convenient to consider the random variables

$$V_i = q_{i-1} U_{i:n} - q_i U_{i-1:n}, \quad i = 1, 2, \ldots, n,$$

where $U_{0:n} \equiv 0$ and $p_i = 1 - q_i = i/(n+1)$, $i = 0, 1, \ldots, n$. Since $\mathbb{E}[U_{i:n}] = p_i$ and $\mathrm{Cov}[U_{i:n}, U_{j:n}] = p_i q_j/(n+2)$, it follows that $\mathbb{E}[V_i] = 1/(n+1) = p_1$, $\mathrm{Var}[V_i] = q_{i-1} q_i/((n+1)(n+2))$ and $\mathrm{Cov}[V_i, V_j] = 0$ for all $i \neq j$, $i, j = 1, 2, \ldots, n$; therefore, the V_i's are uncorrelated random variables. As in (1.1), it can be easily seen that a general linear estimator based on the censored sample has the form

$$L = \sum_{i=1}^{s} c_i^* U^*_{i:n} \stackrel{\mathrm{d}}{=} \theta_1 \sum_{i=1}^{s} c_i^* + \theta_2 \sum_{i=1}^{s} c_i^* U_{i:n} = \theta_1 \sum_{i=1}^{s} c_i^* + \theta_2 \sum_{i=1}^{s} c_i V_i, \tag{5.1}$$

where the constants c_i and c_i^*, $i = 1, 2, \ldots, s$, are related through $c_s^* = c_s q_{s-1}$ and $c_i^* = c_i q_{i-1} - c_{i+1} q_{i+1}$, $i = 1, \ldots, s-1$. Therefore,

$$\mathbb{E}[L] = \theta_1 \sum_{i=1}^{s} c_i^* + \frac{\theta_2}{n+1} \sum_{i=1}^{s} c_i,$$

and thus, L is unbiased for θ_2 iff

$$\sum_{i=1}^{s} c_i^* = 0 \quad \text{and} \quad \sum_{i=1}^{s} c_i = n+1;$$

that is,

$$c_1 = -\frac{n+1}{n} \quad \text{and} \quad \sum_{i=2}^{s} c_i = \frac{(n+1)^2}{n}. \tag{5.2}$$

Using the above notations, we can easily prove the following Theorem. Note that this result is due to Sarhan and Greenberg (1959), but the point here is that for the calculation of BLUE and its variance we do not have to invert a submatrix of $\boldsymbol{\Sigma}$.

Theorem 5.1. *The* BLUE *of* θ_2 *and its variance are given by*

$$L_{\mathrm{U}} = \frac{n+1}{s-1}(U^*_{s:n} - U^*_{1:n}), \ \textit{with} \ \mathrm{Var}[L_{\mathrm{U}}] = \frac{(n+2-s)\theta_2^2}{(n+2)(s-1)}.$$

Proof. Since the V_i's are uncorrelated random variables, it follows from (5.1) and (5.2) that

$$\mathrm{Var}[L] = \frac{\theta_2^2}{(n+1)(n+2)} \sum_{i=1}^{s} c_i^2 q_{i-1} q_i. \tag{5.3}$$

Therefore, minimizing (5.3) with respect to (5.2) by considering the Lagrangian

$$Q(c_2,\ldots,c_s;\lambda)=\sum_{i=2}^{s}c_i^2q_{i-1}q_i-2\lambda\left(\sum_{i=2}^{s}c_i-\frac{(n+1)^2}{n}\right),$$

we get

$$c_i=\frac{\lambda}{q_{i-1}q_i},\quad i=2,\ldots,s,$$

and from (5.2), the Lagrangian multiplier λ simplifies to

$$\lambda=\frac{(n+1)^2}{n}\left(\sum_{i=2}^{s}\frac{1}{q_{i-1}q_i}\right)^{-1}=\frac{n-s+1}{s-1}.$$

Therefore,

$$c_i=\frac{n-s+1}{(s-1)q_{i-1}q_i}=\frac{(n-s+1)(n+1)^2}{(s-1)(n+2-i)(n+1-i)},\quad i=2,\ldots,s,$$

and thus, $c_1^*=-(n+1)/(s-1)$, $c_s^*=(n+1)/(s-1)$ and $c_i^*=0$ for $2<i<s$; this, combined with (5.3), completes the proof. □

By exploiting the above technique (using V_i's), the BLUE of θ_1 (and its variance) can be easily derived as

$$T_{\mathrm{U}}=\frac{1}{s-1}(sU_{1:n}^*-U_{s:n}^*)\quad\text{with } \mathrm{Var}[T_{\mathrm{U}}]=\frac{s\theta_2^2}{(n+1)(n+2)(s-1)}.$$

Moreover, one can easily prove that the above estimators are also *trace-efficient*, *determinant-efficient* and, the stronger, *variance–covariance matrix-efficient* linear unbiased estimators (i.e., $\mathbb{D}[(T,L)']\geqslant\mathbb{D}[(T_{\mathrm{U}},L_{\mathrm{U}})']$ for any linear unbiased estimator (T,L) of (θ_1,θ_2), in the sense that the matrix $\mathbb{D}[(T,L)']-\mathbb{D}[(T_{\mathrm{U}},L_{\mathrm{U}})']$ is non-negative definite); the derivations follow the same arguments as the corresponding ones in Balakrishnan and Rao (1997) for the exponential distribution.

6. Examples and conclusions

Example 6.1 (*Full Sample from the Uniform* $(\theta_1,\theta_1+\theta_2)$ *model*). In this case,

$$\boldsymbol{m}=\frac{1}{n+1}\mathbf{1},\qquad \boldsymbol{D}=\frac{1}{(n+1)^2(n+2)}((n+1)\boldsymbol{I}-\boldsymbol{J}),$$

where $\boldsymbol{I}$ is the $(n-1)$-dimensional identity matrix and $\boldsymbol{J}=\mathbf{1}\mathbf{1}'$ is the $(n-1)$-dimensional matrix with all its elements equal to 1. Therefore,

$$\boldsymbol{D}^{-1}=\tfrac{1}{2}(n+1)(n+2)(2\boldsymbol{I}+\boldsymbol{J}),$$

and from Propositions 2.1 and 2.2 we get the BLUE and the BLIE to be, respectively,

$$L_{\mathrm{U}}=\frac{\boldsymbol{m}'\boldsymbol{D}^{-1}\boldsymbol{Z}^*}{\boldsymbol{m}'\boldsymbol{D}^{-1}\boldsymbol{m}}=\frac{n+1}{n-1}(U_{n:n}^*-U_{1:n}^*),\quad L_{\mathrm{I}}=\frac{\boldsymbol{m}'\boldsymbol{D}^{-1}\boldsymbol{Z}^*}{1+\boldsymbol{m}'\boldsymbol{D}^{-1}\boldsymbol{m}}=\frac{n+2}{n}(U_{n:n}^*-U_{1:n}^*).$$

Observe that

$$\boldsymbol{D}\mathbf{1} = \frac{2}{(n+1)^2(n+2)}\,\mathbf{1} = \frac{2}{(n+1)(n+2)}\,\boldsymbol{m},$$

and Theorem 3.1(iii) (with $\lambda_2 = (n+1)(n+2)/2$) immediately yields that the BLUE (BLIE) is a multiple of the sample range.

Example 6.2 (*Right Censored Sample from the Uniform* $(0, \theta_2)$ *model*). Proceeding as in Section 5, we find the BLUE of θ_2 and its variance to be

$$L_{\mathrm{U}} = \frac{n+1}{s}\,U^*_{s:n} \quad \text{with } \mathrm{Var}[L_{\mathrm{U}}] = \frac{(n+1-s)\theta_2^2}{(n+2)s}.$$

Example 6.3 (*Right Censored Sample from the Uniform* (θ_1, θ_1+1) *model*). Similarly, we find the BLUE of θ_1 and its variance to be

$$T_{\mathrm{U}} = \frac{1}{s-1}\,(sU^*_{1:n} - U^*_{s:n}) \quad \text{with } \mathrm{Var}[T_{\mathrm{U}}] = \frac{s}{(n+1)(n+2)(s-1)}.$$

Example 6.4 (*Right Censored Sample from the Uniform* $(-\theta_2, \theta_2)$ *model*). In this case, we find the BLUE of θ_2 and its variance to be

$$L_{\mathrm{U}} = \frac{n+1}{n^2-n-2-ns+3s}\,(U^*_{s:n} - (n+1-s)U^*_{1:n})$$

with

$$\mathrm{Var}[L_{\mathrm{U}}] = \frac{4(n+1-s)\theta_2^2}{(n+2)(n^2-n-2-ns+3s)}.$$

Example 6.5 (*Bernoulli* (p) *location-scale family with* $p \in (0,1)$ *known*). Assume that $X_1^*, X_2^*, \ldots, X_n^*$ is a random sample from the two valued d.f. assuming probabilities $1-p$ and p (p known) to the unknown reals θ_1 and $\theta_1+\theta_2$, respectively, where $\theta_2 > 0$. In this trivial example, we have

$$\mathbb{P}[Z_1 = \cdots = Z_{n-1} = 0] = \mathbb{P}[\text{all the } X_i\text{'s are equal}] = p^n + (1-p)^n,$$

where $X_1, X_2, \ldots, X_n$ is a random sample from Bernoulli (p) and $Z_i = X_{i+1:n} - X_{i:n}$, $i = 1, \ldots, n-1$, are the corresponding spacings. Observe that $Z_i = Z_i^2$ and $Z_iZ_j = 0$ for all $i \neq j$, $i,j = 1, \ldots, n-1$. Therefore,

$$\mathbb{E}[Z_iR] = \sum_{j=1}^{n-1} \mathbb{E}[Z_iZ_j] = \mathbb{E}[Z_i^2] = \mathbb{E}[Z_i], \quad i = 1, \ldots, n-1,$$

and from Theorem 3.1(ii) (with $\lambda_1 = 1$) we conclude that the BLUE and the BLIE for θ_2 are, respectively,

$$L_{\mathrm{U}} = \frac{1}{1-p^n-(1-p)^n}\,(X^*_{n:n} - X^*_{1:n}) \quad \text{and } L_{\mathrm{I}} = X^*_{n:n} - X^*_{1:n}.$$

Observe that in this trivial case, L_{I} is *always better* than L_{U}, since

$$|L_{\mathrm{I}} - \theta_2| \leqslant |L_{\mathrm{U}} - \theta_2|,$$

and the equality holds iff $X^*_{1:n} = X^*_{n:n}$, an event of probability $p^n + (1-p)^n \to 0$, as $n \to \infty$. This example shows that there are distributions other than the Uniform, such that the BLUE is a constant multiple of the sample range and, therefore, the Uniform is not characterized by this property.

Since the derivations of the above examples are extremely simple, it seems that the method presented in Section 5 is quite effective. Furthermore, it should be noted that the results of Sections 3 and 4 do fairly depend on the representation of the linear location-invariant estimator as a linear function on spacings, indicating the applicability of the presented method. Moreover, as a final observation, we note that the results of Section 2 can be easily applied to any Type-II Censored sample of the form

$$X^*_{i_1:n} \leqslant X^*_{i_2:n} \leqslant \cdots \leqslant X^*_{i_s:n}, \quad 1 \leqslant i_1 < \cdots < i_s \leqslant n \quad (2 \leqslant s \leqslant n),$$

by considering the corresponding spacings

$$\tilde{\boldsymbol{Z}}^* = (\tilde{Z}^*_1, \ldots, \tilde{Z}^*_{s-1})' = (X^*_{i_2:n} - X^*_{i_1:n}, \ldots, X^*_{i_s:n} - X^*_{i_{s-1}:n})',$$

and using formulae (2.4) and (2.10) with $\tilde{\boldsymbol{Z}}^*$, $\tilde{\boldsymbol{m}} = \mathbb{E}[\tilde{\boldsymbol{Z}}]$, $\tilde{\boldsymbol{D}} = \mathbb{D}[\tilde{\boldsymbol{Z}}]$ and $\tilde{\boldsymbol{E}} = \mathbb{E}[\tilde{\boldsymbol{Z}}\tilde{\boldsymbol{Z}}']$ in place of $\boldsymbol{Z}^*$, $\boldsymbol{m}$, $\boldsymbol{D}$ and $\boldsymbol{E}$, respectively, where

$$\tilde{\boldsymbol{Z}} = (\tilde{Z}_1, \ldots, \tilde{Z}_{s-1})' = (X_{i_2:n} - X_{i_1:n}, \ldots, X_{i_s:n} - X_{i_{s-1}:n})'.$$

Therefore, if either $s = 2$ or (4.1) holds, then the conclusion of Theorem 4.1 (that the BLUE of θ_2 is non-negative) remains valid in the general Type-II censored case; for $s = 2$ the BLUE of θ_2 is a multiple of $\tilde{Z}^*_1 = X^*_{i_2:n} - X^*_{i_1:n} = Z^*_{i_1} + \cdots + Z^*_{i_2-1}$, while if $s > 2$ and (4.1) holds then $\tilde{\boldsymbol{m}} > \boldsymbol{0}$ componentwise, and for $k < m$ $(1 \leqslant k < m \leqslant s-1)$ we simply have

$$\mathrm{Cov}[\tilde{Z}_k, \tilde{Z}_m] = \sum_{r=i_k}^{i_{k+1}-1} \sum_{t=i_m}^{i_{m+1}-1} \mathrm{Cov}[Z_r, Z_t] \leqslant 0.$$

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